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# Exact vortex solutions in an extended Skyrme-Faddeev model

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Abstract: We construct exact vortex solutions in $3+1$ dimensions to a theory which is an extension, due to Gies, of the Skyrme-Faddeev model, and that is believed to describe some aspects of the low energy limit of the pure SU(2) Yang-Mills theory. Despite the efforts in the last decades those are the first exact analytical solutions to be constructed for such type of theory. The exact vortices appear in a very particular sector of the theory characterized by special values of the coupling constants, and by a constraint that leads to an infinite number of conserved charges. The theory is scale invariant in that sector, and the solutions satisfy Bogomolny type equations. The energy of the static vortex is proportional to its topological charge, and waves can travel with the speed of light along them, adding to the energy a term proportional to a U(1) Noether charge they create. We believe such vortices may play a role in the strong coupling regime of the pure SU(2) Yang-Mills theory.

Keywords: Integrable Field Theories, Integrable Equations in Physics, Solitons Monopoles and Instantons, Integrable Hierarchies

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The Skyrme-Faddeev (SF) model [1] has been proposed long ago as a four dimensional field theory that can present finite energy knot solitons carrying a topological charge given by the Hopf map $S^3 \to S^2$. Despite many efforts in the last decades no analytical non-trivial solutions have been constructed. The confirmation of that expectation however came some years ago with the construction of numerical solutions employing powerful computer resources [2–4]. The interest in the SF model has increased since then and many applications have been proposed including Bose-Einstein condensates [5] and superconductors [6]. In addition, it has been conjectured [7] that the SF model describes the low energy limit of the pure (no matter) SU(2) Yang-Mills theory, with the knot solitons being interpreted perhaps as glueballs, or some non-trivial vacuum configurations. The conjecture is based on the Cho-Faddeev-Niemi decomposition [7, 8] of the SU(2) Yang-Mills field in terms of an abelian gauge field, a triplet $\vec{n}$ of scalars living on $S^2$ ($\vec{n}^2 = 1$), and two other degrees of freedom which can be expressed by two real scalar fields. There has been several controversies about the validity of that conjecture [9, 10] but a calculation carried out by Gies [11] has indicated that it may indeed be the case provided additional quartic terms are added to the effective Lagrangian.

The purpose of this paper is to present the first exact analytical solutions in $3+1$ dimensions for the extension of the Skyrme-Faddeev model defined by the Lagrangian

$$\mathcal{L} = M^2\, \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} - \frac{1}{e^2}\, \left(\partial_\mu \vec{n} \wedge \partial_\nu \vec{n}\right)^2 + \frac{\beta}{2}\, \left(\partial_\mu \vec{n} \cdot \partial^\mu \vec{n}\right)^2 \tag{1}$$

where $\vec{n}$ is a triplet of real scalar fields taking values on the sphere $S^2$, $M$ is a coupling constant with dimension of mass, $e^2$ and $\beta$ are dimensionless coupling constants. We show that (1) possesses a Bogomolny type sector admitting an infinite number of conserved currents, and when the coupling constants satisfy $\beta\, e^2 = 1$, there exists an infinite class of exact solutions. Among those there are static vortex solutions as well as vortices with waves travelling along them, and enhancing their stability.

The first two terms of (1) correspond to the original SF model [1, 4], and the third one agrees with the term found by Gies in his calculations [11]. A similar extension of the SF model has been considered in [12], but with some important differences. First the model was defined in three space dimensions only and the signs of the second and third terms in (1) were chosen to keep the static Hamiltonian positive definite. The existence of the exact solutions we present in this paper depends crucially on the fact that $e^2$ and $\beta$ have the same sign, and that agrees with the signature of the one-loop Wilsonian effective action for the SU(2) Yang-Mills theory obtained in [11]. Despite of that signature, our solutions belong to a sector of the theory where the energy is positive definite. Solutions for the conformally invariant version of the model, namely the theory (1) without the first term ($M = 0$), have been constructed and will be given in a separate paper [13].

The solutions we construct in this paper can be best explained by using the stereographic projection of $S^2$ in terms of a complex scalar field $u$, i.e.

$$\vec{n} = \left(u + u^*, -i\,(u - u^*)\,, \mid u \mid^2 - 1\right) / \left(1 + \mid u \mid^2\right) \tag{2}$$

One then obtains that

$$\vec{n}\cdot\left(\partial_\mu\vec{n}\wedge\partial_\nu\vec{n}\right) = 2i\,\frac{\left(\partial_\nu u\partial_\mu u^* - \partial_\mu u\partial_\nu u^*\right)}{\left(1+\mid u\mid^2\right)^2}$$
$$\left(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}\right) = 4\,\frac{\partial_\mu u\,\partial^\mu u^*}{\left(1+\mid u\mid^2\right)^2} \tag{3}$$

Therefore, the Lagrangian (1) becomes

$$\mathcal{L} = 4\,M^2\,\frac{\partial_\mu u\,\partial^\mu u^*}{\left(1+\mid u\mid^2\right)^2} + \frac{8}{e^2}\left[\frac{\left(\partial_\mu u\right)^2\left(\partial_\nu u^*\right)^2}{\left(1+\mid u\mid^2\right)^4} + \left(\beta\,e^2-1\right)\,\frac{\left(\partial_\mu u\,\partial^\mu u^*\right)^2}{\left(1+\mid u\mid^2\right)^4}\right] \tag{4}$$

and the Euler-Lagrange equations following from (4), or (1), reads

$$\left(1+\mid u\mid^2\right)\,\partial^\mu\mathcal{K}_\mu - 2\,u^*\,\mathcal{K}_\mu\,\partial^\mu u = 0 \tag{5}$$

together with its complex conjugate, and where[1]

$$\mathcal{K}_\mu \equiv M^2\,\partial_\mu u - \frac{4}{e^2}\,\frac{\left[\left(1-\beta\,e^2\right)\,\left(\partial_\nu u\,\partial^\nu u^*\right)\,\partial_\mu u - \left(\partial_\nu u\partial^\nu u\right)\partial_\mu u^*\right]}{\left(1+\mid u\mid^2\right)^2} \tag{6}$$

It was shown in [16], using a generalization of the zero curvature condition to higher dimensions, that many field theories admit an infinite number of conserved charges when some special constraints are imposed. In theories with target space $S^2$ like (1) the relevant constraint is given by

$$\partial_\mu u\,\partial^\mu u = 0 \tag{7}$$

When (7) is imposed the theory (1) possesses the infinite set of conserved currents given by

$$J_\mu \equiv \frac{\delta G}{\delta u}\,\mathcal{K}^c_\mu - \frac{\delta G}{\delta u^*}\,\mathcal{K}^{c\,*}_\mu \tag{8}$$

where $G$ is any function of $u$ and $u^*$, but not of their derivatives, and $\mathcal{K}^c_\mu$ is obtained from (6) by imposing (7), i.e.

$$\mathcal{K}^c_\mu \equiv M^2\,\partial_\mu u - \frac{4}{e^2}\,\frac{\left(1-\beta\,e^2\right)\,\left(\partial_\nu u\,\partial^\nu u^*\right)\,\partial_\mu u}{\left(1+\mid u\mid^2\right)^2} \tag{9}$$

The conservation of (8) follows from the equations of motion, which now read $\partial^\mu\mathcal{K}^c_\mu = 0$, and the two identities $\mathcal{K}^c_\mu\partial^\mu u = 0$ and $Im\left(\mathcal{K}^c_\mu\partial^\mu u^*\right) = 0$. In the context of SF model those currents were first constructed in [17].

Notice that the constraint (7), in Minkowiski space-time, can be cast into the form

$$\left[\left(\partial_1+i\,\partial_2\right)u\right]\left[\left(\partial_1-i\,\partial_2\right)u\right] = -\left[\left(\partial_3+\partial_0\right)u\right]\left[\left(\partial_3-\partial_0\right)u\right] \tag{10}$$

[1]Notice that this theory has a set of trivial plane wave solutions given by $u = Ae^{i\,k_\mu\,x^\mu}$, with the wave vector satisfying either $k^2=0$ or $k^2 = -\frac{\left(1+A^2\right)^2}{4\,A^2}\,\frac{M^2}{\beta}$. Plane wave solutions for the SF model were considered in [14].

If one now chooses the coupling constants in (1) to satisfy

$$\beta\, e^2 = 1 \tag{11}$$

one gets that eqs. of motion (5) reduce to $\partial^2 u = 0$, or

$$(\partial_1 + i\,\partial_2)\,(\partial_1 - i\,\partial_2)\,u = -\,(\partial_3 + \partial_0)\,(\partial_3 - \partial_0)\,u \tag{12}$$

In a Euclidean space-time the same relations hold by replacing the time coordinate $x^0$ by an imaginary time $-i\,x^4$, and so $\partial_0 \to i\,\partial_4$.

Of course the equations (10) and (12) are solved by field configurations satisfying

$$(\partial_1 + i\,\varepsilon_1\,\partial_2)\,u = 0 \qquad \text{and} \qquad (\partial_3 + \varepsilon_2\,\partial_0)\,u = 0 \tag{13}$$

where $\varepsilon_i = \pm 1$, are signs chosen independently. In addition, equations (13) are satisfied by configurations of the form[2]

$$u = v\,(z)\;w\,(y) \tag{14}$$

where $z = x^1 + i\,\varepsilon_1\,x^2$ and $y = x^3 - \varepsilon_2\,x^0$, with $v\,(z)$ and $w\,(y)$ being arbitrary regular functions of their arguments. Notice that if $u$ satisfies (13), so does any regular functional of it, $\mathcal{F}\,(u)$. Indeed, by taking $\mathcal{F}$ to be the logarithm one observes that the ansatz (14) is mapped into $u = v\,(z) + w\,(y)$. See [19] for similar discussions in $2+1$ dimensions.

Notice that by imposing the constraint (7) and the condition (11) the Lagrangian (4) and equations of motion (5) becomes those of the $CP^1$ model, i.e.

$$\mathcal{L}_{CP^1} = 4\,M^2\,\frac{\partial_\mu u\;\partial^\mu u^*}{(1+\mid u\mid^2)^2} \qquad\qquad \left(1+\mid u\mid^2\right)\,\partial^2 u - 2\,u^*\,\left(\partial^\mu u\right)^2 = 0 \tag{15}$$

Therefore, the solutions of (10) and (12) also solve the $CP^1$ model in $3+1$ dimensions. Therefore, the Skyrme-Faddeev-Gies model (1) shares a common submodel with $CP^1$, defined by the equations $(\partial_\mu u)^2 = 0$, and $\partial^2 u = 0$, together with the condition (11). In fact, the first equation in (13) correspond to the Bogomolny equation, or equivalently the Cauchy-Riemann equations, leading to static solutions of that model in $2+1$ dimensions [20]. The relation between the Bogomolny equation and the constraint (7) was already pointed out in [16] when the generalization of the zero curvature condition was applied to the $CP^1$ model in $2+1$ dimensions.

The structure of the solutions is now clear. The function $v\,(z)$ corresponds to the static lumps of the $CP^1$ model in $2+1$ dimensions. The function $w\,(y)$ corresponds to waves traveling along the $x^3$ direction with the speed of light. In a Euclidean space-time $w\,(y)$ corresponds instead to a second copy of the $CP^1$ lumps. If one wants static solutions in Minkowiski space-time, then it has also to be $x^3$ independent. Therefore, the $CP^1$ lump gets a string like shape in three space dimensions and it corresponds in fact to a static vortex solution for the Skyrme-Faddeev-Gies model (alternatively for the $CP^1$ too, in three space dimensions). One can take $v\,(z)$ to be any finite energy lump solution of

[2]We are grateful to Prof. D. Fairlie for pointing to us that a large class of implicit solutions for the wave equation and the constraint (7) is given in [18], and that they are related to the solutions (14).

the two dimensional $CP^1$ model. For instance, one can take $v \sim z^n$, and $w(y) = 1$ in (14). Using polar coordinates on the $x^1\,x^2$ plane, i.e. $x^1 + i\,\varepsilon_1\,x^2 = \rho\,e^{i\,\varepsilon_1\,\varphi}$, we obtain the static vortex

$$u = \left(\frac{\rho}{a}\right)^n\, e^{i\,\varepsilon_1\,n\,\varphi} \tag{16}$$

where $n$ is an integer, and $a$ an arbitrary parameter with dimension of length. Configurations with several vortices all parallel to the $x^3$-axis can be obtained from the multi lumps of the $CP^1$ model. Numerical and approximate vortex solutions for the SF model have been considered in [14].

One can dress such vortices with waves traveling along the $x^3$ axis. There are several ways of doing it. One way of keeping the energy per unit of length finite is to take $w(y)$ in (14) of the plane wave form, leading to the vortex

$$u = \left(\frac{\rho}{a}\right)^n\, e^{i\left[\varepsilon_1\,n\,\varphi + k\left(x^3 - \varepsilon_2\,x^0\right)\right]} \tag{17}$$

where $k$ is an arbitrary parameter with dimension of $(\text{length})^{-1}$. Vortices with waves have been found in a different context in [15].

If one evaluates the Hamiltonian density for the theory (1) and then imposes the conditions (10) and (11) one obtains

$$\mathcal{H}_c = 4\,M^2\,\frac{\left(\partial_0\,u\,\partial_0\,u^* + \vec{\nabla}u\cdot\vec{\nabla}u^*\right)}{\left(1+\mid u\mid^2\right)^2} \tag{18}$$

where the subindex $c$ means we are imposing (10) and (11). Notice that it coincides with the Hamiltonian density for the $CP^1$ model. We can write that as

$$\begin{aligned}\mathcal{H}_c =\; & \frac{4M^2}{\left(1+\mid u\mid^2\right)^2}\big[\,\mid \partial_1 u + i\varepsilon_1\partial_2 u\mid^2 + \mid \partial_3 u + \varepsilon_2\partial_0 u\mid^2 \\ & + i\varepsilon_1\left(\partial_1 u\partial_2 u^* - \partial_2 u\partial_1 u^*\right) - \varepsilon_2\left(\partial_3 u\partial_0 u^* + \partial_0 u\partial_3 u^*\right)\big] \\ & \geq 4\,M^2\left[i\,\varepsilon_1\frac{\left(\partial_1 u\,\partial_2 u^* - \partial_2 u\,\partial_1 u^*\right)}{\left(1+\mid u\mid^2\right)^2} - \varepsilon_2\frac{\left(\partial_3 u\,\partial_0 u^* + \partial_0 u\,\partial_3 u^*\right)}{\left(1+\mid u\mid^2\right)^2}\right]\end{aligned} \tag{19}$$

The bound on the energy density is clearly saturated by the solutions of (13). In fact, for those solutions one gets

$$\mathcal{H}_c = 8\,M^2\,\left(\mid \partial_z u\mid^2 + \mid \partial_y u\mid^2\right)/\left(1+\mid u\mid^2\right)^2 \tag{20}$$

The same analysis can be done in the Euclidean case, with the same results, replacing the Hamiltonian density by the Lagrangian density, and $x^0 \to -i\,x^4$.

The energy density for the vortex solutions (16) and (17) is then given by[3]

$$\mathcal{H}_c^{\text{vortex}} = 8\,M^2\,\left[\frac{n^2}{\rho^2} + k^2\right]\,\frac{(\rho/a)^{2\,n}}{\left(1+(\rho/a)^{2\,n}\right)^2} \tag{21}$$

[3]Notice that such energy density depends on $\mid n\mid$, and not on the sign of $n$, because $\frac{(\rho/a)^{2\,n}}{\left(1+(\rho/a)^{2\,n}\right)^2} = \frac{(\rho/a)^{-2\,n}}{\left(1+(\rho/a)^{-2\,n}\right)^2}$

Therefore, by integrating on the $x^1x^2$ plane we get the energy per unity lenght. For the static vortex (16) it is

$$\mathcal{E}_{\text{stat.vortex}} = 8\,\pi\, M^2 \;\mid n \mid \tag{22}$$

The integer $n$ is the topological charge associated to the vortex, and defined as the winding number of the map from any circle on the $x^1x^2$ plane, centered at the $x^3$-axis, to the circle $u/\mid u\mid$ on target space. So, the energy per unit length is proportional to the topological charge, a characteristic of solutions saturating the Bogomolny bound.

The energy per unit length for the time-dependent vortex (17) diverges for $n = \pm 1$, and for $\mid n \mid > 1$ it is given by

$$\mathcal{E}_{\text{vortex/wave}} = 8\,\pi\, M^2 \left[\mid n \mid + k^2\, a^2\, I\,(\mid n\mid)\right] \tag{23}$$

where $I\,(n) = \frac{1}{n}\,\Gamma\left(\frac{n+1}{n}\right)\,\Gamma\left(\frac{n-1}{n}\right)$, with $\Gamma$ being the Euler's Gamma function. Notice that $I\,(n)$ is a monotonically decreasing function of $n$ and $I\,(n) \rightarrow 1/n$, as $n \rightarrow \infty$. Therefore, the plane wave contribution to the energy per unit length decreases as $\mid n \mid$ increases.

Notice that a necessary condition for the charge densities $J_0$ associated to (8) to be non-vanishing is that the field $u$ should be time dependent. Therefore, all the charges associated to (8) vanish when evaluated on the static vortex solutions (16). However, the vortices (17), with waves travelling along them, are time dependent solutions and so can have non-vanishing charges associated to (8). Consequently that infinite number of conserved charges introduce selection rules which protect them against decay into lower energy vortex solutions, and so give them some stability. However the question of stability of such solutions under small perturbations is a much more complex issue, and deserves further studies. We should point out however that the static Hamiltonian associated to (1) is positive definite for $M^2 > 0$, $e^2 < 0$, $\beta < 0$ and $\beta\, e^2 \geq 1$. Therefore, our static vortices, saturating the Bogomolny bound (see (19)), have the smallest possible energy in such theory, for that range of coupling constants. That fact makes us believe that for sufficiently large $M^2$, the vortices should be stable under small perturbations.

If one chooses in (8), the functional as $G = -i\,4\,\left(1 + \mid u \mid^2\right)^{-1}$, and if one takes into account (11) one gets that the corresponding conserved current is

$$J_\mu = -i\,4\,M^2\,\frac{\left[u\,\partial_\mu u^* - u^*\,\partial_\mu u\right]}{\left(1 + \mid u\mid^2\right)^2} \tag{24}$$

which corresponds to the Noether current for the theory (1) associated to the symmetry $u \rightarrow e^{i\alpha}\,u$, after the conditions (7) and (11) are imposed. Evaluating the charge per unit length for the solution (17) one gets

$$Q = \int dx^1 dx^2\, J_0 = \varepsilon_2\, 8\,\pi\, M^2\, k\, a^2\, I\,(\mid n\mid) \tag{25}$$

Therefore, the contribution to the energy per unit length (23) coming from the plane wave is proportional to the U(1) Noether charge it gives to the solution.

The spectrum of energy we have got is therefore very similar to that of the so-called Q-lumps constructed in [21] for the 2+1 dimensional $CP^1$ model with a potential. See [22] for more details, [23] for a related model, and also [24] for $Q$-balls solutions on $S^3$. However, the mechanisms involved here are quite different. We have a four dimensional theory, without a potential term depending on $\mid u \mid$ only, and our vortices are not spinning but instead have waves traveling along them with the speed of light. In addition, we again point out that our results are true for the four dimensional $CP^1$ model without a potential, but subjected to the constraint (7).

The theory (1) is invariant under the internal SO(3) symmetry of rotations of the $\vec{n}$ field. It then possesses three conserved Noether currents associated to such symmetry. One of them is that associated to the symmetry $u \to e^{i\alpha} u$, discussed above. However, when the contraint (7) is imposed the reduced model gets an infinite number of conserved currents given by (8). Such currents can not be of the Noether type[4] because the reduced system of equations, namely $\partial^2 u = 0$ and $(\partial_\mu u)^2 = 0$, does not possess a Lagrangian and so the Noether theorem does not apply. Such currents can be understood as hidden symmetries of the reduced model by constructing them using a zero curvature representation as explained in section 6.1.1 of [16]. They are associated to an infinite dimensional non semi-simple Lie algebra made of the semi-direct product of SU(2) with abelian ideals transforming under infinite dimensional representations of SU(2). It is important to understand the role such symmetries may have in the extension of the Skyrme-Faddeev model given by the theory (1).

Another point concerning the enlargement of symmetries is the following. Even though the theory (1) is not scale invariant, the conditions (7) and (11) lead to a sub-model which equations of motion are scale invariant. The condition $(\partial u)^2 = 0$ is conformally invariant but $\partial^2 u = 0$ is only scale invariant in four dimensions.

Finally, we would like to comment on some important points of our construction which relate to the pure SU(2) Yang-Mills theory. Notice that by using the Cho-Faddeev-Niemi decomposition [7, 8] of the SU(2) gauge field $\vec{A}_\mu$, and assuming that at low energies all degrees of freedom get frozen except for those associated to the $\vec{n}$ field, then $\vec{A}_\mu = \partial_\mu \vec{n} \wedge \vec{n}$. Our vortices are solutions of the constraints $\partial_\mu u \partial^\mu u = 0$, and also of $\partial^2 u = 0$. Those equations imply that the field $\vec{n}$ satisfies $\partial^2 \vec{n} + (\partial \vec{n})^2 \, \vec{n} = 0$. Therefore the connection, evaluated on the vortices, satisfies the gauge fixing condition

$$\partial^\mu \vec{A}_\mu = 0 \tag{26}$$

Another point is that, without the use of any condition (equations of motion or constraint), and working in a Euclidean space-time, it follows that (see eq. (3))

$$\vec{A}_\mu \cdot \vec{A}^\mu = \mathcal{H}_c / M^2 \tag{27}$$

where $\mathcal{H}_c$ is given in (18). Therefore as explained in (19), the equations (13) then imply that our vortex solutions minimize locally the quantity $\vec{A}_\mu \cdot \vec{A}^\mu$. In fact, its integral on the plane $x^1 \, x^2$ has the smallest possible value when evaluated on the vortices. Such quantity is used as a gauge fixing condition and it is believed to signal non trivial structures in gauge theories [26].

[4]Except for those three currents associated to the global SO(3) symmetry of (1), contained in (8).



JHEP05(2009)001

In order to have the condition (11) satisfied we need of course the coupling constants $\beta$ and $e^2$ to have the same sign. That implies that the second and third terms in (1) have opposite signs. We are then in a situation which differs from that considered in [12] where the signs of the three terms in (1) were chosen to give positive contributions to the static energy. See also [25] for a discussion on solutions of that type of model. We are however in agreement with the situation in [11]. Indeed, consider the action (1) in four dimensional Euclidean space and take $\beta$ and $e^2$ to be negative. We then have the same sign structure of the effective action given in eq. (14) of [11]. In fact, the condition (11) corresponds to the following relation among the parameters of [11]

$$\ln \frac{k}{\Lambda} = -6\,\pi^2\,\frac{(4-\alpha)}{\alpha\, g^2} \tag{28}$$

where $g^2$ is the gauge coupling constant, $\alpha$ a gauge fixing parameter, $\Lambda$ and $k$ are the UV and IR cutoffs respectively. The values for $\alpha$ in [11] are such that $(4-\alpha)/\alpha$ is positive, and so (28) is indeed consistent with $k < \Lambda$. However, the relation (28) has to be read in its proper context, i.e. the physical limit corresponds to $k \rightarrow 0$, and the perturbative results of [11] may not be applicable in that regime. It would be very interesting to investigate if the condition (11) remains consistent with the renormalization group flows in the IR limit. That could clarify if the solutions calculated in this paper can play a role in the low energy limit of the pure SU(2) Yang-Mills theory.

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## References

[1] L.D. Faddeev, *Quantization of solitons*, Princeton preprint IAS Print-75-QS70 (1975) [SPIRES]; *40 Years in Mathematical Physics*, World Scientific (1995).

[2] R.A. Battye and P.M. Sutcliffe, *To be or knot to be?*, *Phys. Rev. Lett.* **81** (1998) 4798 [hep-th/9808129] [SPIRES].

[3] J. Hietarinta and P. Salo, *Ground state in the Faddeev-Skyrme model*, *Phys. Rev.* **D 62** (2000) 081701 [SPIRES].

[4] L.D. Faddeev and A.J. Niemi, *Knots and particles*, *Nature* **387** (1997) 58 [hep-th/9610193] [SPIRES]; *Toroidal configurations as stable solitons*, hep-th/9705176 [SPIRES].

[5] E. Babaev, L.D. Faddeev and A.J. Niemi, *Hidden symmetry and knot solitons in a charged two- condensate Bose system*, *Phys. Rev.* **B 65** (2002) 100512 [cond-mat/0106152] [SPIRES].

[6] E. Babaev, *Knotted solitons in triplet superconductors*, Phys. Rev. Lett. **88** (2002) 177002 [cond-mat/0106360] [SPIRES].

[7] L.D. Faddeev and A.J. Niemi, *Partially dual variables in* SU(2) *Yang-Mills theory*, Phys. Rev. Lett. **82** (1999) 1624 [hep-th/9807069] [SPIRES].

[8] Y.M. Cho, *A restricted gauge theory*, Phys. Rev. **D 21** (1980) 1080 [SPIRES]; *Extended gauge theory and its mass spectrum*, Phys. Rev. **D 23** (1981) 2415 [SPIRES].

[9] L. Dittmann, T. Heinzl and A. Wipf, *A lattice study of the Faddeev-Niemi effective action*, Nucl. Phys. Proc. Suppl. **106** (2002) 649 [hep-lat/0110026] [SPIRES]; *Effective theories of confinement*, Nucl. Phys. Proc. Suppl. **108** (2002) 63 [hep-lat/0111037] [SPIRES].

[10] L.D. Faddeev, *Knots as possible excitations of the quantum Yang-Mills fields*, arXiv:0805.1624 [SPIRES].

[11] H. Gies, *Wilsonian effective action for* SU(2) *Yang-Mills theory with Cho-Faddeev-Niemi-Shabanov decomposition*, Phys. Rev. **D 63** (2001) 125023 [hep-th/0102026] [SPIRES].

[12] J. Gladikowski and M. Hellmund, *Static solitons with non-zero Hopf number*, Phys. Rev. **D 56** (1997) 5194 [hep-th/9609035] [SPIRES].

[13] L.A. Ferreira and A.C. Risério do Bonfim, *Self-dual Hopfions*, in preparation.

[14] M. Hirayama, C.-G. Shi and J. Yamashita, *Elliptic solutions of the Skyrme model*, Phys. Rev. **D 67** (2003) 105009 [hep-th/0303092] [SPIRES];
M. Hirayama and C.-G. Shi, *A class of exact solutions of the Faddeev model*, Phys. Rev. **D 69** (2004) 045001 [hep-th/0310042] [SPIRES];
C.-G. Shi and M. Hirayama, *Approximate vortex solution of Faddeev model*, Int. J. Mod. Phys. **A 23** (2008) 1361 [arXiv:0712.4330] [SPIRES];
J. Hietarinta, J. Jaykka and P. Salo, *Relaxation of twisted vortices in the Faddeev-Skyrme model*, Phys. Lett. **A 321** (2004) 324 [cond-mat/0309499] [SPIRES].

[15] M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, 1/2*,* 1/4 *and* 1/8 *BPS equations in SUSY Yang-Mills-Higgs systems: field theoretical brane configurations*, Nucl. Phys. **B 752** (2006) 140 [hep-th/0506257] [SPIRES];
M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Solitons in the Higgs phase: the moduli matrix approach*, J. Phys. **A 39** (2006) R315 [hep-th/0602170] [SPIRES].

[16] O. Alvarez, L.A. Ferreira and J. Sanchez Guillen, *A new approach to integrable theories in any dimension*, Nucl. Phys. **B 529** (1998) 689 [hep-th/9710147] [SPIRES].

[17] H. Aratyn, L.A. Ferreira and A.H. Zimerman, *Toroidal solitons in* $3+1$ *dimensional integrable theories*, Phys. Lett. **B 456** (1999) 162 [hep-th/9902141] [SPIRES]; *Exact static soliton solutions of* $3+1$ *dimensional integrable theory with nonzero Hopf numbers*, Phys. Rev. Lett. **83** (1999) 1723 [hep-th/9905079] [SPIRES].

[18] D.B. Fairlie, *Implicit solutions to some Lorentz invariant non-linear equations revisited, J. Nonlinear Math. Phys.* **12** (2005) 449 [math-ph/0412005].

[19] R.S. Ward, *Slowly moving lumps in the* $C^1$ *model in (2+1)-dimensions*, Phys. Lett. **b 158** (1985) 424 [SPIRES].

[20] A.M. Polyakov and A.A. Belavin, *Metastable states of two-dimensional isotropic ferromagnets*, *JETP Lett.* **22** (1975) 245 [*Pisma Zh. Eksp. Teor. Fiz.* **22** (1975) 503] [SPIRES].

[21] R.A. Leese, *Q lumps and their interactions*, *Nucl. Phys.* **B 366** (1991) 283 [SPIRES].

[22] N.S. Manton and P. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge U.K. (2004), pg. 493 [SPIRES].

[23] B.M. A.G. Piette, B.J. Schroers and W.J. Zakrzewski, *Multi-solitons in a two-dimensional Skyrme model*, *Z. Phys.* **C 65** (1995) 165 [hep-th/9406160] [SPIRES].

[24] C. Adam, J. Sanchez-Guillen and A. Wereszczynski, *Hopf solitons and Hopf Q-balls on $S^3$*, *Eur. Phys. J.* **C 47** (2006) 513 [hep-th/0602008] [SPIRES].

[25] N. Sawado, N. Shiiki and S. Tanaka, *Hopf soliton solutions from low energy effective action of* SU(2) *Yang-Mills theory*, *Mod. Phys. Lett.* **A 21** (2006) 1189 [hep-ph/0511208] [SPIRES].

[26] F.V. Gubarev, L. Stodolsky and V.I. Zakharov, *On the significance of the quantity "A sqared"*, *Phys. Rev. Lett.* **86** (2001) 2220 [hep-ph/0010057] [SPIRES].